Job Mobility of Couples: Shapley-Scarf Markets with Coalitional Trade

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A Quick overview

• Growing literature on the engineering of school choice (2-sided matching) with several theoretical studies of situations involving externalities:


→ externalities among couples, peer effects, student assignment problem with neighbors

• But very little about trade, or re-assignment: job mobility

→ why? The convincing top trading cycle algorithm (Gale): implements a competitive equilibrium, Core stable, Pareto if no indifference, and strategy-proof
• *The TTC algorithm:* 5 jobs, 5 individuals, individual \(i\) assigned to job \(i\)

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 4 & 1 & 1 & 4 \\
3 & 5 & 4 & 3 & 2 \\
4 & 3 & 5 & 5 & 1 \\
\ldots & \ldots & \ldots & \ldots & 3
\end{pmatrix}
\]

Row 1 cycle: 1 \(\rightarrow\) 2 \(\rightarrow\) 4 \(\rightarrow\) 1: remove 1,2,4 from the picture

\[
\begin{pmatrix}
1_{out} & 2_{out} & 3 & 4_{out} & 5 \\
. & . & . & . & . \\
. & . & . & . & . \\
. & 5 & . & . & . \\
\ldots & \ldots & \ldots & \ldots & 3
\end{pmatrix}
\]

Final cycle: 3 \(\rightarrow\) 5 \(\rightarrow\) 3

\(\sigma = (2, 4, 5, 1, 3)\) competitive allocation, Pareto, and Core stable
• Shapley-Scarf market (Shapley, Scarf 1974)
  - $n$ indivisible goods (set $J$), $n$ individuals (set $I$)
  - allocation $\sigma = \text{bijection } I \to J$
  - $\sigma^0 = \text{initial allocation}$
  - individual $i \mapsto \text{ranking of goods } \succ_i$

• Interpretation:
  - goods = jobs
  - trade = job mobility scheme

• Our goal:
  - investigating properties of centralized job mobility procedures involving couples
  - centralized: individual submit preferences over jobs
  - externalities among partners $\rightarrow$ what happens to my partner matters!
  - 2 types of externality: distance (please let us be close to each other) and love (please my partner as well as myself)
JOB MOBILITY OF COUPLES: WHEN DISTANCE MATTERS

• Fixed marriage structure: partition of $I$ into couples and singles

• Fixed geographical structure: partition of jobs into locations

• Individuals submit preferences over jobs

• Individuals compare allocations according to the quality of their own job and the distance to partner: comparing preferences does not resume to comparing jobs
Preferences over allocations

- $i \rightarrow$ linear order $\succ_i$ over $\{1, \ldots, n\} \times \{0, 1\}$
  - discrete metric
    - $(k, 0) =$ job with rank $k$ at distance 0 to partner,
    - $(k, 1) =$ job with rank $k$ at distance 1 to partner

- **Responsiveness:**
  - $k < k' \Rightarrow (k, x) \succ_i (k', x)$ for all $x$
  - $(k, 0) \succ_i (k, 1)$ for all $k$

- Trade-off between job quality and distance
  - rank $k \rightarrow$ switch $\phi(k) = \max\{1 \leq t \leq n : (t, 0) \succ_i (k, 1)\}$

- 2 polar cases:
  - **Job oriented** preferences: $\phi(k) = k$ for all $k$
  - **Distance oriented** preferences: $\phi(k) = n$ for all $k$
**Question**: can one design a centralized procedure with Core stable outcome?

- **CORE C**: allocation $\sigma$ is Core stable if the initial allocation is not unanimously preferred by any couple to $\sigma$ (weak couple rationality), and there no coalition $S$ can be formed such that
  - it does not break couples: $C = \{i, i'\}, i \in S \Rightarrow i' \in S$
  - all $S$ members are strictly better off by reallocating jobs within $S$

- **STRONG CORE $C^S$**: Same with couple rationality replaced by standard individual rationality
Example

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 2 & 4 & 4 & 6 & 5 \\
6 & 4 & 6 & 2 & 1 & 4 \\
1 & 3 & 3 & 5 & 3 & 2 \\
3 & 5 & 1 & 3 & 5 & 6 \\
... & ... & ... & ... & ... & ...
\end{pmatrix}
\]

couples \{1, 2\}, \{3, 4\}, \{5, 6\} Locations \(L_1 = \{1, 3, 5\}\), \(L_2 = \{2, 4, 6\}\)

\(\sigma = (1, 3, 2, 4, 5, 6)\), \(\sigma' = (6, 2, 3, 4, 1, 5)\)

\(S = \{1, 2, 5, 6\}\)

\[
\begin{pmatrix}
\sigma & (3, 1) & (3, 1) & (4, 1) & (4, 1) \\
\sigma' & (2, 0) & (1, 0) & (2, 0) & (1, 0) \\
\end{pmatrix}
\]

\(\rightarrow \sigma\) is not (strong) Core stable
**MAIN RESULTS**

1 The Core may be empty under responsive preferences

2 The strong Core is non empty under job oriented preferences: standard TTC works!

3 Distance oriented preferences without singles: the geography matters
   - If all couples can be made close (0-solvability), the Core is always non-empty, while the strong Core may be empty. based on an algorithm that fails strategy-proofness
   - If at least one couple (or more) must stay distant, the Core may be empty

4 Distance-oriented preferences with singles: the disturbing latin lover
   - Even under 0-solvability, the Core may be empty
JOB MOBILITY OF COUPLES: THE POINT SYSTEM

- Point system rather widespread in real-life procedures:
  - individuals are entitled to get as new position only those for which they accumulated enough points
  - point results from history of career

- Similar in spirit to the price mechanism
  - (much) simplifying assumption: value in points of current job = 'income', and 'price' of a job = number of points required to apply for it
  - ignores a lot of interesting strategic issues
  - but captures the fact that point levels reflect tensions in demand (do you like Brittany?)

- How does the price mechanism behave in presence of externalities in couple preferences?
Modelling the price mechanism

• Critical issues: joint versus individual job applications (preferences)? Transferability of points among partners?

• $I = \{1, \ldots, i, \ldots n\} =$ set of individuals
  $J = \{1, \ldots, j, \ldots n\} =$ set of jobs

• Exogenous partition $C$ of $I$ into couples $C = \{i, i\}'\}$

• job $j \rightarrow$ price $p_j$

• allocation $= \text{bijection } \sigma : I \rightarrow J$
  initial allocation $\sigma^0$

• $i \rightarrow$ weak order $R_i$ over allocations
• TWO TYPES OF BUDGET CONSTRAINT
\((\sigma^0, p, C) \rightarrow \textbf{Strong budget set } B_S^C(\sigma^0, p)\)
\[= \{\sigma : p_\sigma(i) \leq p_\sigma^0(i) \text{ and } p_\sigma(i') \leq p_\sigma^0(i')\}\]
\[\textbf{Weak budget set } B_W^i(\sigma^0, p)\]
\[= \{\sigma : p_\sigma(i) + p_\sigma(i') \leq p_\sigma^0(i) + p_\sigma^0(i')\}\]

• THREE TYPES OF EQUILIBRIUM \((\sigma, p)\)
  - Selfish \(E^{self} \): \(\sigma \in \arg \max_{B_i(p)} R_i\) for all \(i\)
  - Cooperative \(E^{coop} \): \(\sigma\) is Pareto efficient in \(B_C(\sigma^0, p)\) for all \(C\)
  - Coordinated \(E^{coor} \): \(\sigma\) implements a Nash equ. in \(B_C(\sigma^0, p)\) for all \(C\)

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<tr>
<th></th>
<th>selfish</th>
<th>cooperative</th>
<th>coordinated</th>
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<tbody>
<tr>
<td>strong</td>
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Preference domains

- **Coalition-Responsiveness** CR: each partner $i$ has a ranking $p_i$ of jobs, and any unilateral improvement of one partner’s situation is beneficial for the other.

$$jp_i j' \Rightarrow (j, k) \succ_i (j', k) \text{ and } (j, k) \succ_i (j', k)$$

- **Weak lexicographic property** WL: facing any subset of available jobs $H \subseteq J$, each partner has a unique job in $H$ that she wants the couple to be endowed with. Who gets that job depends on what the other partner receives.

$$H = \{1, 2, 3, 4\}, \ (1, 4) \succ_i (4, 2) \succ_i (3, 4) \succ_i \ldots$$

and $$(3, 4) \succ_i (1, 3) \succ_i (3, 2) \succ_i \ldots$$
### MAIN RESULTS

- **Set-comparison of equilibria**

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<thead>
<tr>
<th>strong, weak</th>
<th>coop</th>
<th>coord</th>
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<tbody>
<tr>
<td>self</td>
<td>$\subseteq (\emptyset, WL, CR)$</td>
<td>$\subseteq (\emptyset, WL)$</td>
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<tr>
<td>coop</td>
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<td>$\emptyset$ if CR $\emptyset$ if WL</td>
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| strong self  | $\emptyset$, $\notin$ (CR) | weak self       |
| strong coop  | $\emptyset$, $\notin$ (CR) | weak coop       |
| strong coord | $\subseteq (\emptyset, WL, CR)$ | weak coord      |
- *Properties of equilibria: individual rationality, Pareto optimality*

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<td><strong>coop</strong></td>
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<td>strong: <em>no</em></td>
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<td><em>yes</em> if joint pref</td>
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- *Properties of equilibria: Core stability*

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<td><strong>self</strong></td>
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• Existence of strong equilibria

- If preferences are not responsive, a strong selfish equilibrium may not exist

- The domain of responsive preferences is maximal for the existence of strong selfish equilibria

⇒ strong selfish, cooperative and coordinated equilibria always exist under responsiveness

key argument: standard TTC works, but equilibria are no longer necessarily outcomes of TTC. And TTC is not strategy-proof
- **Existence of weak equilibria**

- If preferences are responsive, a weak selfish equilibrium may not exist

- Same if preferences are identical

- The domain of weakly lexicographic preferences is maximal for the existence of weak cooperative equilibria

- If preferences are weakly lexicographic and responsive, a weak coordinated equilibrium always exists

key argument: a new algorithm ... not strategy-proof
CONCLUSION

- very preliminary steps towards a proper treatment of externalities in reassignment procedures

- shows the wide range and equilibrium and stability concepts, and the critical role played by preferences (and initial conditions) for existence

- many pending issues, e.g.
  - existence of strategy-proof Core stable mechanisms for job mobility
  - characterization and non-cooperative foundations of equilibrium allocations